Modular Arithmetic

1 Preliminary Definitions

Definition (A divides B)**.** Let $A, B \in \mathbb{Z}$. We say that A *divides* B (or A is a *divisor* of B), denoted $A|B$, if there is a number $C \in \mathbb{Z}$ such that $B = AC$.

Definition (Prime number)**.** Let $P \in \mathbb{N}$. We say that P is a *prime number* if $P \geq 2$ and the only divisors of P are 1 and P.

Definition (Congruence modulo N)**.** Let A, B and N be positive integers. We denote by $A \bmod N$ the remainder you get when you divide A by N. Note that $A \bmod N \in \{0, 1, 2, \ldots, N-1\}$. We say that A and B are congruent modulo N, denoted $A \equiv_N B$ (or $A \equiv B \bmod N$), if $A \bmod N = B \bmod N$.

Exercise (A characterization for congruence modulo N)**.** Show that $A \equiv_N B$ if and only if $N|(A - B)$.

Solution. If $A \equiv_N B$, then by definition, A and B have the same remainder R when they are divided by N. So we can write $A = Q \cdot N + R$ for some $Q \in \mathbb{Z}$ and $B = Q' \cdot N + R$ for some $Q' \in \mathbb{Z}$. Then $A - B = (Q - Q') \cdot N$, and therefore $N|(A - B)$.

Suppose $N|(A - B)$. Write $A = Q \cdot N + R$ for some $Q \in \mathbb{Z}$ and $R \in \{0, 1, \ldots, N-1\}$. Also write $B = Q' \cdot N + R'$ for some $Q' \in \mathbb{Z}$ and $R' \in \{0, 1, \ldots, N-1\}$. Then $A - B = (Q - Q') \cdot N + (R - R')$. Since $N|(A - B)$, it must be the case that $N|(R - R')$. Since $R - R'$ is an integer between $-(N-1)$ and $(N-1)$, the only way N can divide $R - R'$ is if $R = R'$, i.e., $A \equiv_N B$. ■

Note. The above characterization of $A \equiv_N B$ can be taken as the definition of $A \equiv_N B$. Indeed, it is used a lot in proofs.

Definition (gcd)**.** We write $\gcd(A, B)$ to denote the greatest common divisor of A and B. Note that for any A, $\gcd(A, 1) = 1$ and $\gcd(A, 0) = A$.

Definition (Relatively prime)**.** We say that A and B are relatively prime if $\gcd(A, B) = 1$.

In the section below, we first give the definitions of basic modular operations like addition, subtraction, multiplication, division and exponentiation. We also explore some of their properties. In the section after, we look at the computational complexity of these operations. Being able to compute these operations efficiently is crucial for applications.

2 Modular operations: Definitions and properties

2.1 Addition and subtraction

Definition ($\mathbb{Z}_N$)**.** We let $\mathbb{Z}_N$ denote the set $\{0, 1, 2, \ldots, N-1\}$.

Definition (Addition in $\mathbb{Z}_N$)**.** For $A, B \in \mathbb{Z}_N$, we define the *addition* of A and B, denoted $A +_N B$, as $(A + B) \bmod N$. When N is clear from the context, we can drop the subscript N from $+_N$ and write $+$. For $N = 5$, we can represent the addition operation in $\mathbb{Z}_5$ using the following table.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

In $\mathbb{Z}_N$, the element 0 is called the *additive identity*. It has the property that for any $A \in \mathbb{Z}_N$, $A +_N 0 = 0 +_N A = A$.

Exercise (Addition modulo N behaves nicely)**.**

- Show that if $A \equiv_N B$ and $A' \equiv_N B'$, then $A + A' \equiv_N B + B'$.
- Show that for any $A, B \in \mathbb{Z}$,

$$(A + B) \bmod N = (A \bmod N) +_N (B \bmod N).$$

Solution. Part 1: Since $A \equiv_N B$, we have $N|(A - B)$, and since $A' \equiv_N B'$, we have $N|(A' - B')$. This implies $N|(A - B) + (A' - B')$, or in other words, $N|(A + A') - (B + B')$. And this is equivalent to $A + A' \equiv_N B + B'$.
Part 2: Follows from the previous part and the definition of $+_N$. ■

Definition (Additive inverse)**.** Let $A \in \mathbb{Z}_N$. The *additive inverse* of A, denoted $-A$, is defined to be an element in $\mathbb{Z}_N$ such that $A +_N -A = 0$.

Exercise (Additive inverses modulo N are unique)**.** Show that every element of $\mathbb{Z}_N$ has a unique additive inverse.

Solution. The additive inverse of $A \in \mathbb{Z}_N$ is 0 if $A = 0$, and is $N - A$ otherwise.

To show uniqueness, assume that $-A$ and $-A'$ are both additive inverses of A. Then $A +_N -A = A +_N -A' = 0$, which implies $-A = -A'$. ■

Definition (Subtraction in $\mathbb{Z}_N$)**.** Let $A, B \in \mathbb{Z}_N$. We define "A minus B", denoted $A -_N B$, as $A +_N -B$.

Exercise (Addition table permutation property)**.** Show that in the addition table of $\mathbb{Z}_N$, every row and column is a permutation of the elements $\mathbb{Z}_N$.

Solution. We argue that every row contains distinct elements of $\mathbb{Z}_N$, which implies that every row is a permutation of $\mathbb{Z}_N$. Take an arbitrary row, which corresponds to some element $A \in \mathbb{Z}_N$. Suppose for the sake of contradiction that two entries of this row are the same. Then there exists B and B' in $\mathbb{Z}_N$, $B \neq B'$, such that $A +_N B = A +_N B'$. But then if we add $-A$ to both sides of the equality, we get $B = B'$, a contradiction.

The argument for the columns is the same. ■

2.2 Multiplication and division

Definition (Multiplication in $\mathbb{Z}_N$)**.** For $A, B \in \mathbb{Z}_N$, we define the *multiplication* of A and B, denoted $A \cdot_N B$, as $AB \bmod N$. If N is clear from the context, we can drop the subscript N from $\cdot_N$ and write $\cdot$. Furthermore, we can even drop $\cdot$ and represent $A \cdot_N B$ as simply AB.

Exercise (Multiplication modulo N behaves nicely)**.**

- Show that if $A \equiv_N B$ and $A' \equiv_N B'$, then $AA' \equiv_N BB'$.
- Show that for any $A, B \in \mathbb{Z}$,

$$AB \bmod N = (A \bmod N) \cdot_N (B \bmod N).$$

Solution. Part 1: Hint: Write the elements as $QN + R$ for some $Q \in \mathbb{Z}$ and remainder $R \in \{0, 1, \dots, N-1\}$.

Part 2: Follows from Part 1 and the definition of $\cdot_N$. ■

Definition (Multiplicative inverse)**.** Let $A \in \mathbb{Z}_N$. The *multiplicative inverse* of A, denoted A^{-1}, is defined to be an element in $\mathbb{Z}_N$ such that $A \cdot_N A^{-1} = 1$.

Proposition (Multiplicative inverse characterization)**.** *Let $A, N \in \mathbb{N}$. The multiplicative inverse of A in $\mathbb{Z}_N$ exists if and only if* $\gcd(A, N) = 1$.

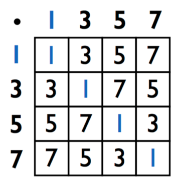
Definition (Division in $\mathbb{Z}_N$)**.** Let $A, B \in \mathbb{Z}_N$, where B has a multiplicative inverse B^{-1}. Then we define "A divided by B", denoted $A/_N B$, as $A \cdot_N B^{-1}$.

Definition ($\mathbb{Z}_N^*$)**.** We let $\mathbb{Z}_N^*$ denote the set $\{A \in \mathbb{Z}_N : \gcd(A, N) = 1\}$. In other words, $\mathbb{Z}_N^*$ is the set of all elements of $\mathbb{Z}_N$ that have a multiplicative inverse.

Exercise ($\mathbb{Z}_N^*$ is closed under multiplication)**.** Show that $\mathbb{Z}_N^*$ is *closed* under multiplication, i.e., $A, B \in \mathbb{Z}_N^* \implies A \cdot_N B \in \mathbb{Z}_N^*$.

Solution. If A and B are in $\mathbb{Z}_N^*$, then they have inverses $A^{-1}, B^{-1} \in \mathbb{Z}_N^*$. To show $A \cdot_N B$ is in $\mathbb{Z}_N^*$, we need to show that it has an inverse modulo N. And indeed, $B^{-1} \cdot_N A^{-1}$ is the inverse of $A \cdot_N B$ since $A \cdot_N B \cdot_N B^{-1} \cdot_N A^{-1} = 1$. ■

Note (Multiplication table for $\mathbb{Z}_N^*$)**.** Similar to an addition table for $\mathbb{Z}_N$, one can consider a multiplication table for $\mathbb{Z}_N^*$. For example, $\mathbb{Z}_8^* = \{1, 3, 5, 7\}$, and the multiplication table is as below:



•	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Exercise (Multiplication table permutation property)**.** Show that in the multiplication table of $\mathbb{Z}_N^*$, every row and column is a permutation of the elements $\mathbb{Z}_N^*$.

Solution. We argue that every row contains distinct elements of $\mathbb{Z}_N^*$, which implies that every row is a permutation of $\mathbb{Z}_N^*$. Take an arbitrary row, which corresponds to some element $A \in \mathbb{Z}_N^*$. Suppose for the sake of contradiction that two entries of this row are the same. Then there exists B and B' in $\mathbb{Z}_N^*$, $B \neq B'$, such that $A \cdot_N B = A \cdot_N B'$. But then if we multiply both sides of the equality by A^{-1} we get $B = B'$, a contradiction.

The argument for the columns is the same. ■

Definition (Euler totient function)**.** The Euler totient function $\varphi : \mathbb{N} \to \mathbb{N}$ is defined as $\varphi(N) = |\mathbb{Z}_N^*|$. By convention, $\varphi(0) = 0$.

Exercise ($\varphi(P)$ and $\varphi(PQ)$)**.** Show that for P a prime number, $\varphi(P) = P - 1$. Also show that for P and Q distinct prime numbers, $\varphi(PQ) = (P-1)(Q-1)$.

Solution. We know that $\varphi(N)$ is the number of elements in $\{0, 1, 2, \ldots, N-1\}$ that are relatively prime to N. If P is a prime number, then for any $A \in \{1, 2, \ldots, P-1\}$, we have $\gcd(A, P) = 1$. And $\gcd(0, P) = P$. So $\varphi(P) = P - 1$.

When $N = PQ$ for distinct primes P and Q, we need to determine the number of elements in $\{0, 1, 2, \ldots, PQ-1\}$ that are relatively prime to PQ. The elements that are **not** relatively prime to PQ are 0 and

$$P, 2P, 3P, \ldots, (Q-1)P,$$
$$Q, 2Q, 3Q, \ldots, (P-1)Q.$$

In total there are $1 + (Q-1) + (P-1) = Q + P - 1$ of these elements. Then the number of elements that *are* relatively prime to PQ is $PQ - (Q + P - 1) = PQ - Q - P + 1 = (P-1)(Q-1)$. ■

2.3 Exponentiation

Definition (Exponentiation in $\mathbb{Z}_N$)**.** Let $A \in \mathbb{Z}_N$ and $E \in \mathbb{Z}$. We write A^E to denote

$$\underbrace{A \cdot_N A \cdot_N \cdots \cdot_N A}_{E \text{ times}}.$$

Theorem (Euler's Theorem)**.** *For any $A \in \mathbb{Z}_N^*$, $A^{\varphi(N)} = 1$. Equivalently, for any $A, N \in \mathbb{Z}$ with $\gcd(A, N) = 1$, $A^{\varphi(N)} \equiv_N 1$.*

Proof. (In this proof we drop the subscript N from the multiplication notation.) Take an arbitrary $A \in \mathbb{Z}_N^*$. Let $B_1, B_2, \ldots, B_k$ be the elements of $\mathbb{Z}_N^*$, where $k = \varphi(N)$. By Exercise (Multiplication table permutation property), $\{AB_1, AB_2, \ldots, AB_k\} = \mathbb{Z}_N^*$. The product of all the elements in the first set can be written as $(AB_1)(AB_2)\cdots(AB_k)$. This must be equal to the product $B_1 B_2 \cdots B_k$, i.e.

$$(AB_1)(AB_2)\cdots(AB_k) = B_1 B_2 \cdots B_k.$$

Dividing both sides by $B_1 B_2 \cdots B_k$ (i.e. multiplying both sides by the inverse of $B_1 B_2 \cdots B_k$), we get $A^k = 1$, as desired. □

Note (Fermat's Little Theorem)**.** When N is a prime number, then Euler's Theorem is known as Fermat's Little Theorem.

Exercise (Reducing exponent modulo $\varphi(N)$)**.** Let $A \in \mathbb{Z}_N^*$ and $E \in \mathbb{Z}$. Show that $A^E \equiv_N A^{E \bmod \varphi(N)}$.

Solution. We can write $E = \varphi(N) \cdot Q + R$ where Q is some integer and $R \in \{0, 1, \ldots, \varphi(N) - 1\}$ is the remainder. So $E \equiv_{\varphi(N)} R$. Then

$$A^E = A^{\varphi(N)\cdot Q+R} = \left(A^{\varphi(N)}\right)^Q \cdot A^R = A^R,$$

where for the last equality, we used Theorem (Euler's Theorem). ■

Exercise (Modular computation by hand)**.** Compute by hand $102^{98} \bmod 7$.

Solution. Since $\gcd(102, 7) = 1$, we can use the previous exercise and reduce the exponent modulo 6. So $102^{98} \equiv_7 102^2$. Furthermore, 102 can be reduced modulo 7. So $102^2 \equiv_7 4^2$. And 16 modulo 7 is 2, which is the answer. ■

Important (Exponent lives in $\mathbb{Z}_{\varphi(N)}$)**.** What the previous two exercises demonstrate is that if we are exponentiating an element $A \in \mathbb{Z}_N^*$, then we can effectively think of the exponent as living in the set $\mathbb{Z}_{\varphi(N)}$. This will be important to keep in mind when we cover Cryptography later.

Definition (Generator in $\mathbb{Z}_N^*$)**.** Let $A \in \mathbb{Z}_N^*$. We say that A is a *generator* if

$$\{A^E : E \in \mathbb{Z}_{\varphi(N)}\} = \mathbb{Z}_N^*.$$

Theorem ($\mathbb{Z}_P^*$ contains a generator)**.** *If P is a prime number, then $\mathbb{Z}_P^*$ contains a generator.*

3 Modular operations: Computational complexity

In this section, we will look at the computational complexities of doing the basic modular operations discussed in the previous section. We will use the fact that addition, subtraction, multiplication and division operations can be computed efficiently in $\mathbb{Z}$. Note that $A \bmod N$ is easy to compute by dividing A by N and seeing what the remainder is.

3.1 Addition and subtraction

In order to compute $A +_N B$ in $\mathbb{Z}_N$, we can simply add A and B in $\mathbb{Z}$ and then take the sum modulo N. To compute $A -_N B$, we can do $A + (N - B)$ in $\mathbb{Z}$ and then take the result modulo N.

3.2 Multiplication and division

In order to compute $A \cdot_N B$ in $\mathbb{Z}_N$, we can multiply A and B in $\mathbb{Z}$ and then take the product modulo N. To compute $A /_N B = A \cdot_N B^{-1}$, we first need to figure out whether B has a multiplicative inverse. Recall that B^{-1} exists if and only if B and N are relatively prime, i.e. $\gcd(B, N) = 1$. The following algorithm, known as *Euclid's Algorithm*, efficiently computes the greatest common divisor of two numbers.

Algorithm (Euclid's gcd algorithm)**.** $\gcd(A, B)$:

- If $B = 0$, then return A.
- Else return $\gcd(B, A \bmod B)$.

Exercise (gcd property)**.** Show that if $A \geq B$, $\gcd(A, B) = \gcd(A - B, B)$. Use this to show that Euclid's Algorithm correctly computes the greatest common divisor of two numbers.

Solution. If x divides A and B, then it must divide $A-B$. In particular, $\gcd(A,B)$ divides both B and $A-B$, and therefore $\gcd(A-B,B) \geq \gcd(A,B)$.

If x divides $A-B$ and B, then it must divide A. In particular, $\gcd(A-B,B)$ divides both A and B, and therefore $\gcd(A,B) \geq \gcd(A-B,B)$.

So we can conclude $\gcd(A,B) = \gcd(A-B,B)$.

When $A \geq B$, if we iteratively use the equality $\gcd(A,B) = \gcd(A-B,B)$ to subtract B from the larger number A, then we will eventually arrive at $\gcd(A,B) = \gcd(A \bmod B, B)$. For example:

$$\begin{aligned}\gcd(6004,6) &= \gcd(5998,6)\\ &= \gcd(5992,6)\\ &= \gcd(5986,6)\\ &\cdots\\ &= \gcd(4,6).\end{aligned}$$

So $\gcd(6004,6)$ eventually ends up at $\gcd(6004 \bmod 6, 6)$.

$\gcd(A \bmod B, B)$ is obviously equal to $\gcd(B, A \bmod B)$. So one can show that Euclid's algorithm is correct by an induction argument, where the base case corresponds to the base case of the recursive algorithm, and the induction step corresponds to the recursive call. ■

Exercise (Time complexity of Euclid's algorithm)**.** Suppose A and B can be represented with at most n bits each. Give an upper bound on the number of recursive calls Euclid's Algorithm makes in terms of n.

Solution. We claim that $A \bmod B \leq A/2$. To see this, we case on whether $A \geq 2B$. If $A \geq 2B$, then the claim is true because $A \bmod B$ is always less than B. If $A < 2B$, then the claim is true because $A \bmod B = A - B < A - A/2 = A/2$.

Now when we call the algorithm with input (A,B) and make a recursive call, the next pair of inputs is $(B, A \bmod B)$. Using the claim above, we have $(A \bmod B) \cdot B \leq AB/2$. So in each recursive call, the product of the inputs is going down by a factor of at least 2. And this implies the number of recursive calls is at most $\log_2(AB)$, which is $O(n)$ if A and B are at most n-bits. ■

Using Euclid's Algorithm, we can check if $\gcd(B,N) = 1$ and determine if B has a multiplicative inverse. It turns out that a slight modification of Euclid's Algorithm also allows us to compute B^{-1} if it exists. In order to show this, we first need a definition.

Definition (Miix)**.** Let $A,B,C \in \mathbb{N}$. We say that C is a *miix* of A and B if

$$C = kA + \ell B$$

for some $k, \ell \in \mathbb{Z}$.

Exercise (Miix vs gcd 1)**.** Let $A,B,C \in \mathbb{N}$. Show that if C is a miix of A and B then C is a multiple of $\gcd(A,B)$.

Solution. Let $G = \gcd(A,B)$. If C is a miix of A and B, then $C = kA + \ell B$. Furthermore, we know we can write $A = xG$ for some integer x, and $B = yG$ for some integer y. Therefore

$$C = kA + \ell B = kxG + \ell yG = G(kx + \ell y),$$

which shows C is a multiple of G. ■

Exercise (Miix vs gcd 2)**.** Show how to extend Euclid's algorithm so that it outputs k and ℓ such that $\gcd(A,B) = kA + \ell B$. Then conclude that if C is any multiple of $\gcd(A,B)$, then C is a miix of A and B.

Solution. Here is the extended Euclid's algorithm.

```
def Extended-Euclid$(A, B)$ :
1. If $B$ divides $A$, return $(B, 0, 1)$.
2. Else:
3.     $(G, k, \ell) =$ Extended-Euclid$(B, A \bmod B)$
4.     Return $(G, \ell, (k - \ell \cdot \lfloor A/B \rfloor))$
```

Here, we have modified Euclid's algorithm to return a tuple of variables; Extended-Euclid$(A, B) = (G, k, \ell)$ where $G = \gcd(A, B)$ and $G = k \cdot A + \ell \cdot B$. By the correctness of Euclid's algorithm, we can conclude that the returned value G is the gcd (both algorithms do the same calculations for G). We use induction on the number of steps to argue correctness of the returned values (k, ℓ).

Base Case: If B divides A, the algorithm correctly returns $k = 0$ and $\ell = 1$.

Induction Step: Suppose the algorithm ran for $s > 1$ steps. By the induction hypothesis, we can assume that $G = k \cdot B + \ell \cdot (A \bmod B)$. Since we can write $A = q \cdot B + (A \bmod B)$ where $q = \lfloor A/B \rfloor$, we can say $G = k \cdot B + \ell(A - q \cdot B)$. Thus, the returned tuple $(G, \ell, k - \ell \cdot \lfloor A/B \rfloor)$ is correct.

The above algorithm shows that $\gcd(A, B)$ is a miix of A and B. So if C is a multiple of $\gcd(A, B)$, it is also a miix of A and B. ■

Suppose B has a multiplicative inverse modulo N, i.e. $\gcd(B, N) = 1$. Then by the previous exercise, we can obtain k and ℓ such that $1 = kB + \ell N$. If we take this equation modulo N, we get that $kB \equiv_N 1$. Therefore k is the multiplicative inverse of B.

To sum up, if we want to compute $A /_N B = A \cdot_N B^{-1}$, we can first compute B^{-1} and then compute $A \cdot_N B^{-1}$.

Exercise. Prove Proposition (Multiplicative inverse characterization) using the previous two exercises.

Solution. The previous two exercises imply that C is a miix of A and B if and only if C is a multiple of $\gcd(A, B)$. Then,

$$\begin{aligned}
A^{-1} \text{ exists} &\iff \exists k \text{ such that } kA \equiv_N 1 \\
&\iff \exists k \text{ such that } N \text{ divides } kA - 1 \\
&\iff \exists k, q \text{ such that } kA - 1 = qN \\
&\iff \exists k, q \text{ such that } 1 = kA + (-q)N \\
&\iff 1 \text{ is a miix of } A \text{ and } N \\
&\iff \gcd(A, N) = 1.
\end{aligned}$$

■

3.3 Exponentiation

Given $N \in \mathbb{N}$, $A \in \mathbb{Z}_N$ and $E \in \mathbb{N}$, we can compute $A^E \bmod N$ efficiently. Assume that A, E and N can be represented using at most n bits each. The algorithm below is known as *fast modular exponentiation*. To understand how the algorithm works, see the example following the algorithm.

Algorithm (Fast modular exponentiation)**.** FME(A, E, N):

- Repeatedly square A to obtain
  $A^2 \bmod N$, $A^4 \bmod N$, $A^8 \bmod N$, ..., $A^{2^n} \bmod N$.

- Multiply together (modulo N) the powers of A so that the product is A^E.
  To figure out which powers to multiply, look at the binary representation of E.

Consider the example of computing $2337^{53} \bmod 100$. The first step of the algorithm computes

$$\begin{gathered} 2337^2 \bmod 100 \\ 2337^4 \bmod 100 \\ 2337^8 \bmod 100 \\ 2337^{16} \bmod 100 \\ 2337^{32} \bmod 100 \end{gathered}$$

by squaring 2337 modulo 100 5 times. The binary representation of 53 is 110101. This implies that

$$53 = 1 + 4 + 16 + 32.$$

Therefore to calculate $2337^{53} \bmod 100$, the second step of the algorithm does:

$$(2337 \bmod 100) \cdot (2337^4 \bmod 100) \cdot (2337^{16} \bmod 100) \cdot (2337^{32} \bmod 100).$$

Exercise. Suppose A, E and N are integers that can be represented using at most n bits. Give an upper bound on the running time of the above algorithm in terms of n.

3.4 Taking roots

Consider the following computational problem. You are given $A, E, N \in \mathbb{N}$ with $A \in \mathbb{Z}_N^*$. The goal is to output $B \in \mathbb{N}$ such that $B^E \equiv_N A$. In other words, the goal is to find the E'th root of A in $\mathbb{Z}_N^*$ (if it exists). Many experts believe (but cannot prove) that this problem cannot be computed in polynomial time. The assumed hardness of this problem is used in the famous RSA cryptosystem (see the chapter on Cryptography for details).

3.5 Taking logarithms

Consider the following computational problem which is known as the Discrete Log Problem in $\mathbb{Z}_P^*$. You are given $A, B, P \in \mathbb{N}$, where P is a prime number, $A \in \mathbb{Z}_P^*$, and $B \in \mathbb{Z}_P^*$ is a generator (Theorem ($\mathbb{Z}_P^*$ contains a generator)) tells us that $\mathbb{Z}_P^*$ always contains a generator). The goal is to output $X \in \mathbb{N}$ such that $B^X \equiv_P A$. In a sense, this is like trying to compute $\log_B A$ in $\mathbb{Z}_N^*$. Many experts believe (but cannot prove) that this problem cannot be computed in polynomial time. The assumed hardness of this problem is used in the famous Diffie-Hellman secret key exchange protocol (see the chapter on Cryptography for details).

4 Check Your Understanding

Problem.

1. Wilson's theorem states that N is prime if and only if $(N-1)! \equiv_N N-1$. Can we use this fact in the obvious way to design a polynomial time algorithm for isPrime?
2. True or false: Suppose that $A^K \equiv_N 1$, where $A \neq 1$. Then $\varphi(N)|K$.
3. Determine, with proof, $750^{10002} \pmod{251}$. Note that 251 is prime.
4. Determine, with proof, $251^{(15^{251})} \pmod{7}$.
5. Let G be a generator in $\mathbb{Z}_N^*$, and let $R \in \mathbb{Z}_{\varphi(N)}$ be chosen uniformly at random. For every $A \in \mathbb{Z}_N^*$, determine $\mathbf{Pr}[G^R = A]$.

6. Let $R \in \mathbb{Z}_N^*$ be chosen uniformly at random. For every $A, B \in \mathbb{Z}_N^*$, determine $\mathbf{Pr}[A \cdot_N R = B]$.

7. What is Euler's Theorem?

8. What is a generator in $\mathbb{Z}_N^*$?

9. Consider the addition, subtraction, multiplication, division, exponentiation, taking logarithm, and taking root. Which of these operations are known to be polynomial-time solvable when the universe is the integers? Which of them are known to be polynomial-time solvable in the modular universe?

10. What is an efficient algorithm for finding the multiplicative inverse of an element in $\mathbb{Z}_N^*$?

5 High-Order Bits

Important. 1. The theory behind modular arithmetic should mostly be review. It is important to know how the basic operations are defined and what kind of properties they have.

2. For which modular arithmetic operations do we have an efficient algorithm for (and what is that algorithm)? For which operations do we not expect to have an efficient algorithm? And for these operations, why don't the "obvious" algorithms work? You should have a solid understanding of all these. Please internalize Important Note (**??**) and Important Note (**??**).